

A NON-LINEAR DYNAMICAL SYSTEMS' PROOF OF KRAFT-MCMILLAN INEQUALITY AND ITS CONVERSE

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In this short paper, we shall provide a dynamical systems' proof of the famous Kraft-McMillan inequality and its converse. Kraft-McMillan inequality is a basic result in information theory which gives a necessary and sufficient condition for the lengths of the codewords of a code to be uniquely decodable [1, 2, 3].

1 Kraft-McMillan Inequality

Given a binary prefix code set C for an alphabet set A , the codewords c_1, c_2, \dots, c_N with lengths l_1, l_2, \dots, l_N necessarily satisfy:

$$\sum_{i=1}^N 2^{-l_i} \leq 1 \quad (1)$$

where $N = |A|$, the cardinality of set A . A binary prefix code C is a set of binary codewords such that no codeword is a prefix of another. Prefix codes are known to be uniquely decodable and easy to decode. A famous example of prefix codes are the Huffman codes which have minimum redundancy.

The Binary map

Consider the binary map (Fig. 1) $T : [0, 1) \rightarrow [0, 1)$:

$$\begin{aligned} x &\mapsto 2x, & 0 \leq x < \frac{1}{2} \\ &\mapsto 2x - 1, & \frac{1}{2} \leq x < 1. \end{aligned}$$

It is well known that the binary map is a non-linear chaotic dynamical system, which preserves the Lebesgue measure (ordinary length measure). We shall prove two simple lemmas regarding the binary map which will be used to prove the Kraft-McMillan's inequality.

Lemma 1:

Given any sequence (or string) S of 0s and 1s of length m , there exists a unique interval on the binary map of length 2^{-m} such that all initial conditions in that interval will have S as the binary symbolic sequence corresponding to the first m iterations.

Proof:

Consider the given string S of length m as a binary prefix in $[0, 1)$ (i.e. think of S as $0.S$ in binary). The interval $[0.S\overline{0}, 0.S\overline{1})$, where the overline indicates infinite repetition, consists of all possible binary numbers in $[0, 1)$ which have S as the desired prefix. All these binary numbers when fed as initial conditions to the binary map will yield S as the symbolic sequence in m iterations (this is because the binary map can be thought of as a shift map which spits out the leading bits of the binary representation of the initial condition). The length of this interval is $0.S\overline{1} - 0.S\overline{0}$ which is 2^{-m} . \square

Lemma 2:

Two symbolic sequences S_1 and S_2 which are not prefixes of each other correspond to two **disjoint** intervals of lengths 2^{-m_1} and 2^{-m_2} respectively, where m_1 and m_2 are the lengths of S_1 and S_2 respectively.

Proof:

The proof is obvious. \square

Proof of Kraft-McMillan Inequality

Since c_1, c_2, \dots, c_N with lengths l_1, l_2, \dots, l_N are prefix codes, using Lemma 1 and 2, these can be seen as symbolic sequences of **disjoint** intervals on $[0, 1)$ with lengths $2^{-l_1}, 2^{-l_2}, \dots, 2^{-l_N}$ respectively. Any collection of disjoint intervals on $[0, 1)$ necessarily satisfy Equation 1. \square

2 Converse of Kraft-McMillan inequality

Given a set of codeword lengths that satisfy Equation 1, there exists a uniquely decodable binary prefix code with these codeword lengths.

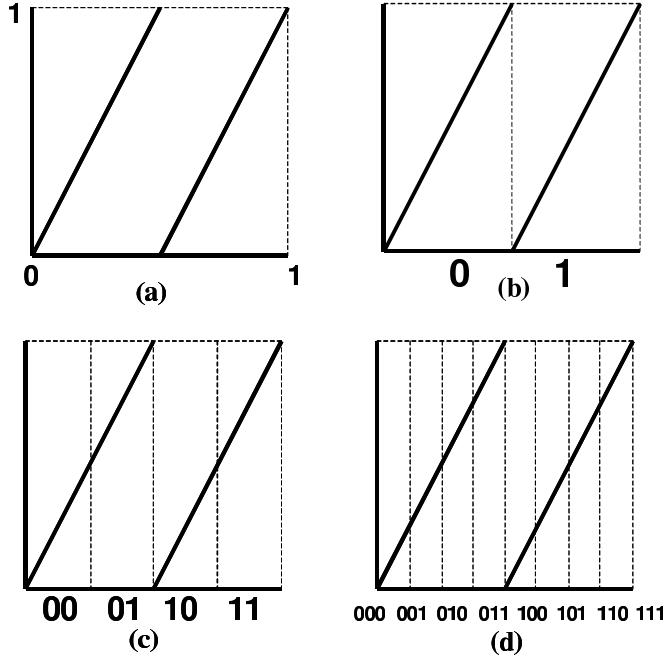


Figure 1: (a) Binary Map. (b) Symbolic sequences of length 1. (c) Symbolic sequences of length 2. (d) Symbolic sequences of length 3.

Proof:

Let l_1, l_2, \dots, l_M be the specified **distinct** codeword lengths such that they satisfy Equation 1. Without loss of generality, let us assume that $l_1 < l_2 < \dots < l_M$. Let there be a_1 codewords of length l_1 , a_2 codewords of length l_2 and so on up to a_M codewords of length l_M . Kraft-McMillan inequality can be re-written as:

$$\sum_{i=1}^M a_i 2^{-l_i} \leq 1, \quad \sum_{i=1}^M a_i = N. \quad (2)$$

where $N = |A|$ as before. Let us determine the maximum number of codewords that can have a particular codeword length l_i while still satisfying Equation 2. If there are $2^{l_i} + 1$ or more codewords with length l_i , then $(2^{l_i} + 1)2^{-l_i} = 1 + 2^{-l_i} > 1$ violating Equation 2. Thus there can at most be 2^{l_i} codewords of length l_i .

Let us begin with l_1 . We know that there are exactly 2^{l_1} disjoint intervals with length 2^{-l_1} on the binary map which have symbolic sequence of length l_1 . Since the intervals are disjoint, the symbolic sequences are necessarily prefix codewords. We first assign the symbolic sequences of a_1 of these disjoint intervals as codewords. Once a_1 disjoint intervals of length 2^{-l_1} were used up, we have lost $a_1 2^{l_2-l_1}$ intervals of length 2^{-l_2} . The number of available disjoint intervals of length 2^{-l_2} is $2^{l_2} - a_1 2^{l_2-l_1}$. If $a_2 < 2^{l_2} - a_1 2^{l_2-l_1}$ then we can allocate disjoint intervals to a_2 codewords of length l_2 . This requires $a_2 < 2^{l_2}(1 - a_1 2^{-l_1})$, which reduces to $a_1 2^{-l_1} + a_2 2^{-l_2} < 1$ which is necessarily true from Equation 2. Thus we can use the symbolic sequence of a_2 disjoint intervals as prefix codewords (of length

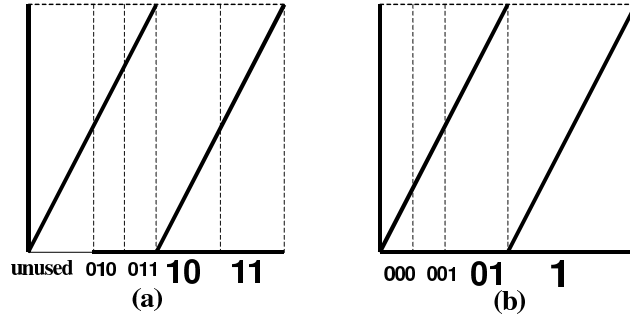


Figure 2: Converse of Kraft-McMillan inequality. (a) Assigning intervals to codewords of lengths $\{3, 3, 2, 2\}$ which satisfies the Kraft-McMillan inequality. (b) Assigning intervals to codewords of lengths $\{3, 3, 2, 1\}$ which satisfies Kraft-McMillan inequality with equality. This is known as a *complete* code.

l_2). This argument is repeated for a_3 and so on until we have allocated unique disjoint intervals to all codewords (see example in Fig. 2). We have thus proved the converse of Kraft-McMillan inequality by construction of prefix codewords using symbolic sequences of disjoint intervals on the binary map. \square

The arguments above can be extended in a straightforward manner for ternary and higher bases. In the case of codewords of base- B , the B -ary dynamical system is used ($x \mapsto Bx \bmod 1$ for all $x \in [0, 1)$).

References

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